Hypercomplex Numbers, Associated Metric Spaces, and Extension of Relativistic Hyperboloid

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Abstract

We undertake to develop a successful framework for commutative-associative hypercomplex numbers with the view to explicate and study associated geometric and generalized-relativistic concepts, basing on an interesting possibility to introduce appropriate multilinear metric forms in the treatment. The scalar polyproduct, which extends the ordinary scalar product used in bilinear (Euclidean and pseudo-Euclidean) theories, has been proposed and applied to be a generalized metric base for the approach. A fundamental concept of multilinear isometry is proposed. This renders possible to muse upon various relativistic physical applications based on anisotropic *versus* ordinary spatially-rotational case.

1. INTRODUCTION

William Hamilton's discovery of quaternions in 1843 was the first event of history when the concept of complex numbers has been successfully generalized. Although the quaternions, as being comprised of the ordered quadruples of real numbers possessing a convenient noncommutative multiplication law, were playing a historically important part in development of both the algebra and the geometry, they were the methods of vector analysis which really streamlined powerful applications. As a farther extension of quaternions, the so-called octaves were developed after Cayley, - a striking particular feature was occured to be an absence of any associativity in the involved multiplications. Recently, many articles in the hypercomplex literature were focused on the adoption of quaternionic methods to Dirac equation and to quantum mechanics (see the works [1-23] and references therein). The polynumbers to be dealt with, like ordinary complex or bicomplex numbers, share many important properties of ordinary real numbers, including commutability, associativity, distributivity, existence of zero and unity; but also has important differences, namely, presence of divisors of zero and absence of ordering.

It proves possible to define the concept of a polyform to serve as a norm for members of the polynumber algebra, which in turn makes us develop a geometry for such the algebra by assigning lengths to associated vectors by means of values of the respective radicals of the polyform. Remarkably, the geometric structure of the space can be extended even farther on the basis of the symmetrical multilinear form, a scalar polyproduct in a sense, which can be proposed to generalize the polyform. Implying possible relativistic physical applications, we shall restrict our treatment to the dimension $N = 4$, so that the quadrahyperbolic numbers will be the subject.

The new facet that has been opened in this vein is the metric relationship with the Finsler-Geometry presupposition. Namely, when we assign the length to a polynumber-associated vector by means of the 4-th radical of the 4-th degree polyform, we get the length which obeys the property of homogeneity of degree one. The latter property, however, is a corner-stone for the Finsler Geometry and for its modern relativistic applications [24-29]. Therefore, there can be arranged a one-to-one correspondence between the set of quadrahyperbolic numbers and the vector space of a Finsler type. In this way, the equation of an Finslerian indicatrix is tantamount to the expression for the modulus of a unit quadrahyperbolic number. In a sense, the respective Finsler space acquires a qualitatively new property: its vectors can be not only summed or deducted, but also multiplied, and even divided (though the last operation is unambiguous only if the length of a divisor differs from zero).

If it is possible to indicate a basis such that the squares of the basis elements become equal $+1$, 0, or -1, then one can conventionally says that the algebra is of a *square-type* nature. Such polynumbers may be of hyperbolic, elliptic, parabolic, or mixed type. In the hyperbolic case the squares of all basis units are equal to $+1$, in the elliptic case there is at least one unity which square is -1, and in the parabolic case there is at least one member for which the square is zero. For mixed polynumbers, basis units of each type may occur. The ordinary real, complex, double, and dual numbers are particular cases of square-type polynumbers. It is natural to expect that each interesting class of polynumbers possesses a powerful geometric representation in an n-dimensional vector space.

The comparison between the ordinary hyperboloid and our $\mathcal{I}_4$-surface clearly watersheds between the pseudo-Euclidean traditional framework and the present $\mathcal{H}_4$-approach: the anisotropic space-time is set forth $\mathcal{VS}$ the spatially-isotropic idealization. Actually, the $\mathcal{H}_4$-approach starts with stipulation that the total space-time manifold exhibits four

distinguished directions, - which can be matched to the fact that the real physical world exhibit many anisotropies as judged on local as well on cosmic scales. For distribution of matter and energy in the Universe is far from being isotropic. Then the $\mathcal{H}_4$-approach may serve the purpose to reflect geometrically and algebraically the anisotropic circumstances. In this vein the set $1, I, J, K$ acquires the algebraic interpretation to mean the hyperunits which corresponds to the anisotropic directions.

The work has been structured as follows.

We shall deal mainly with the four-dimensional case, which is of the particular importance for physical relativistic applications. In Section 2, we introduce initial definitions for respective quadrahyperbolic numbers, emphasizing the existence of attractive definitions for the mutual conjugates, the polyform, and the norm. The introduced concept of mutual conjugacy, though being similar to that applicable in case of complex, double, and dual numbers, does not correspond to the concept of conjugacy used for quaternions and octaves, - for the latters are not polynumbers in our sense because their multiplication is noncommutative. Many aspects and representations get simplified significantly on using the absolute basis which is formed by the divisors of zero. Section 2 is finished with the exponent representation for the quadrahyperbolic numbers under study, which involves for them the exponential arguments. In Section 3, the consideration is transgressed into the isomorphic vector space, whence the implications of associated metric forms are of our particular attention. Using the forms, we propose in Section 4 the fundamental concepts of the transversality, the orthogonality, and the angle for the quadrahyperbolic numbers under study. In particular, a due generalization of the Pythagorean theorem is derived. In Section 5, we propose necessary definitions to generalize concepts of arcs, triangles, and cones, and also introduce the fundamental notion of isometry.

Is it possible to alternate the theory of ordinary complex holomorphic functions, developed on the basis of the Cauchy-Riemann equations, to get a successful holomorphic hypercomplex-valued functions adapted to the quadrahyperbolic numbers? A due positive answer to the question can be found in Section 6. The Finslerian topics covered briefly in Section 7. In the last Section 8, basic points of our approach and relevant out-looks will be emphasized. In the appended section Research Problems we shall list some directions in which one can see subject develop.

2. Basic Definitions

We introduce *quadrahyperbolic numbers*

$$A = a'_1 1 + a'_2 I + a'_3 J + a'_4 K, \tag{2.1}$$

where a'_p are real numbers called *the components*, and $1, I, J, K$ are *basis units*. Given another member

$$B = b'_1 1 + b'_2 I + b'_3 J + b'_4 K, \tag{2.2}$$

the sum and the product of the two such numbers are respectively defined as

$$C = A + B \tag{2.3}$$

with

$$C = (a'_1 + b'_1)1 + (a'_2 + b'_2)I + (a'_3 + b'_3)J + (a'_4 + b'_4)K, \tag{2.4}$$

and as

$$D = AB \tag{2.5}$$

with

$$D = (a_1'b_1' + a_2'b_2' + a_3'b_3' + a_4'b_4')1 + (a_1'b_2' + a_2'b_1' + a_3'b_4' + a_4'b_3')I$$

$$+ (a_1'b_3' + a_2'b_4' + a_3'b_1' + a_4'b_2')J + (a_1'b_4' + a_2'b_3 + a_3'b_2' + a_4'b_1')K, \tag{2.6}$$

whenever the multiplication table of basis units is

	1	I	J	K
1	1	I	J	K
I	I	1	K	J
J	J	K	1	I
K	K	J	I	1

(2.7)

It follows from this table that

$$I^2 = J^2 = K^2 = 1. \tag{2.8}$$

Whence we obtain the algebra of commutative and associative hypercomplex numbers, in which the multiplication table allows to qualify the corresponding hypercomplex numbers as *hyperbolic*, to be referred to as $\mathcal{H}_4$-numbers. Under these conditions, we say that we deal with *the quadrahyperbolic algebra*, to be denoted as $\mathcal{AH}_4$.

We shall proceed by the following important

DEFINITION. Members of a set A_1, A_2, A_3, A_4 are called *mutually conjugate*, if the symmetric polynomials

$$A_1 + A_2 + A_3 + A_4 = P_1, \tag{2.9}$$

$$A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4 = P_2, \tag{2.10}$$

$$A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4 = P_3, \tag{2.11}$$

and

$$A_1A_2A_3A_4 = P_4 \tag{2.12}$$

are real.

We also introduce

DEFINITION. The product P_4 given by (2.12) is called *the polyform* of the number $A \in \mathcal{AH}_4$, with $A \equiv A_1$.

Next, it is possible to define the concept of *the quadrahyperbolic number's modulus*

$$|A| = \sqrt[4]{|A_1A_2A_3A_4|}\,. \tag{2.13}$$

This satisfies the ordinary properties of a modulus:

$$|\lambda A| = |\lambda||A| \tag{2.14}$$

and

$$|AB| = |A||B|, \tag{2.15}$$

where λ is a real number; A and B are two hypercomplex numbers.

Given a number $A \in \mathcal{AH}_4$, it is easy and attractive to propose the following explicit set of mutual conjugates:

$$A_1 = a_1' + a_2'I + a_3'J + a_4'K, \tag{2.16}$$

$$A_2 = a'_1 - a'_2 I + a'_3 J - a'_4 K, \tag{2.17}$$

$$A_3 = a'_1 + a'_2 I - a'_3 J - a'_4 K, \tag{2.18}$$

$$A_4 = a'_1 - a'_2 I - a'_3 J + a'_4 K, \tag{2.19}$$

where $A_1 \equiv A$. The direct calculations yield

$$A_1 + A_2 + A_3 + A_4 = 4a'_1, \tag{2.20}$$

$$A_1A_2 + A_1A_3 + A_1A_4 + A_2A_3 + A_2A_4 + A_3A_4 = 6{a'_1}^2 - 2{a'_2}^2 - 2{a'_3}^2 - 2{a'_4}^2, \tag{2.21}$$

$$A_1A_2A_3 + A_1A_2A_4 + A_1A_3A_4 + A_2A_3A_4 = 4{a'_1}^3 - 4a'_1{a'_2}^2 - 4a'_1{a'_3}^2 - 4a_1{a'_4}^2 + 8a'_2a'_3a'_4, \tag{2.22}$$

$$A_1A_2A_3A_4 = {a'_1}^4 + {a'_2}^4 + {a'_3}^4 + {a'_4}^4 - 2{a'_1}^2{a'_2}^2 - 2{a'_1}^2{a'_3}^2 - 2{a'_1}^2{a'_4}^2$$

$$- 2{a'_2}^2{a'_3}^2 - 2{a'_2}^2{a'_4}^2 - 2{a'_3}^2{a'_4}^2 + 8a'_1a'_2a'_3a'_4. \tag{2.23}$$

The fact that the product of mutual conjugates is a real number implies the possibility to define *the operation of division*, treated as an operation inverse to multiplication. So the inverse A^{-1} of a given number A is the number

$$A^{-1} \stackrel{\text{def}}{=} \frac{A_2A_3A_4}{P_4}. \tag{2.24}$$

Inverses exist only for those numbers whose polyform P_4, as given by (2.12), differs from zero. If a number is not zero, but it's polyform value is equal to zero, then the number is called a *divisor of zero*.

In the algebra $\mathcal{AH}_4$, we can propose the following four explicit divisors of zero

$$S_1 = \frac{1}{4}(1 + I + J + K), \tag{2.25}$$

$$S_2 = \frac{1}{4}(1 - I + J - K), \tag{2.26}$$

$$S_3 = \frac{1}{4}(1 + I - J - K), \tag{2.27}$$

$$S_4 = \frac{1}{4}(1 - I - J + K) \tag{2.28}$$

with the simple algebraic properties: $S_iS_j = S_i$, if $i = j$, and $S_iS_j = 0$, if $i \neq j$. We will call such divisors *absolute*, and the basis consisting of them will be called an *absolute basis.*

Inversely, the units $1, I, J, K$ can readily be expressed through the divisors

$$1 = S_1 + S_2 + S_3 + S_4, \tag{2.29}$$

$$I = S_1 - S_2 + S_3 - S_4, \tag{2.30}$$

$$J = S_1 + S_2 - S_3 - S_4, \tag{2.31}$$

$$K = S_1 - S_2 - S_3 + S_4. \tag{2.32}$$

Numbers from $\mathcal{AH}_4$, when written with respect to the absolute basis, can be easily multiplied and divided. Indeed, the product of two numbers A and B will be

$$AB = (a_1b_1)S_1 + (a_2b_2)S_2 + (a_3b_3)S_3 + (a_4b_4)S_4 \tag{2.33}$$

and their quotient will read

$$\frac{A}{B} = \frac{a_1}{b_1}S_1 + \frac{a_2}{b_2}S_2 + \frac{a_3}{b_3}S_3 + \frac{a_4}{b_4}S_4. \tag{2.34}$$

Usage of the absolute basis reveals the remarkable fact that the structure of the algebra $\mathcal{AH}_4$ is isomorphic to the algebra of real diagonal matrices, for the set of mutual conjugates (see (2.9)-(2.12)) takes on the form

$$A_1 = a_1S_1 + a_2S_2 + a_3S_3 + a_4S_4, \tag{2.35}$$

$$A_2 = a_2S_1 + a_1S_2 + a_4S_3 + a_3S_4, \tag{2.36}$$

$$A_3 = a_3S_1 + a_4S_2 + a_1S_3 + a_2S_4, \tag{2.37}$$

$$A_4 = a_4S_1 + a_3S_2 + a_2S_3 + a_1S_4. \tag{2.38}$$

Therefore, the polyform of a number in the absolute basis reads merely as

$$P_4 = a_1a_2a_3a_4, \tag{2.39}$$

and the simple expression

$$|A| = \sqrt[4]{|a_1a_2a_3a_4|} \tag{2.40}$$

is obtained for the modulus of a quadrahyperbolic number.

Various functions can be introduced on the set of polynumbers $A \in \mathcal{AH}_4$ by the help of series. First of all, the exponential function can be be defined as

$$e^X = 1 + X + \frac{X^2}{2} + \ldots, \tag{2.41}$$

where X is an arbitrary polynumber. The associated logarithmic function can be given quite traditionally:

$$\ln X = \text{inverse of } (e^X). \tag{2.42}$$

Also,

$$\cosh X = 1 + \frac{X^2}{2!} + \ldots, \qquad \sinh X = X + \frac{X^3}{3!} + \ldots. \tag{2.43}$$

Any number $A = a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4$, whenever the components a_p are all greater than zero with respect to the absolute basis, can be represented by the following *exponential form*:

$$A = |A| e^{\alpha I + \beta J + \gamma K}, \tag{2.44}$$

$$|A| = e^{\delta}, \tag{2.45}$$

where $|A|$ is the modulus (2.40); the real numbers α, β, and γ, similarally as in case of complex and double numbers, may be called *the arguments* of a quadrahyperbolic number A; the real number δ represents the modulus. This entails

$$\alpha = \frac{1}{4} \ln \frac{a_1 a_3}{a_2 a_4} = \frac{1}{4}(\ln a_1 - \ln a_2 + \ln a_3 - \ln a_4), \tag{2.46}$$

$$\beta = \frac{1}{4} \ln \frac{a_1 a_2}{a_3 a_4} = \frac{1}{4}(\ln a_1 + \ln a_2 - \ln a_3 - \ln a_4), \tag{2.47}$$

$$\gamma = \frac{1}{4} \ln \frac{a_1 a_4}{a_2 a_3} = \frac{1}{4}(\ln a_1 - \ln a_2 - \ln a_3 + \ln a_4), \tag{2.48}$$

$$\delta = \frac{1}{4} \ln(a^1 a^2 a^3 a^4) = \frac{1}{4}(\ln a^1 + \ln a^2 + \ln a^3 + \ln a^4), \tag{2.49}$$

where $\ln x$ is the ordinary logarithmic function of a real argument x. Inversely,

$$\ln a^1 = \delta + \alpha + \beta + \gamma, \tag{2.50}$$

$$\ln a^2 = \delta - \alpha + \beta - \gamma, \tag{2.51}$$

$$\ln a^3 = \delta + \alpha - \beta - \gamma, \tag{2.52}$$

$$\ln a^4 = \delta - \alpha - \beta + \gamma. \tag{2.53}$$

Since each imaginary unit satisfies the hyperbolic analog of Euler's formula:

$$e^{\alpha I} = \cosh \alpha + I \sinh \alpha, \tag{2.54}$$

the exponent of an arbitrary quadrahyperbolic number $X = \delta + \alpha I + \beta J + \gamma K$ can be given as

$$e^X = (\cosh \delta + \sinh \delta)(\cosh \alpha + I \sinh \alpha)(\cosh \beta + J \sinh \beta)(\cosh \gamma + K \sinh \gamma), \tag{2.55}$$

where $\cosh x$ and $\sinh x$ are ordinary hyperbolic cosine and sine, respectively.

3. Associated Quadrahyperbolic Vector Space $\mathcal{VH}_4$

The algebra $\mathcal{AH}_4$ can be juxtaposed by *the vector space* $\mathcal{VH}_4$ in a quite obvious way. Having assumed a fixed basis in $\mathcal{VH}_4$, we shall denote respective images of $\mathcal{H}_4$-numbers by bolds:

$$\mathbf{A} \sim A, \quad \mathbf{A} \in \mathcal{VH}_4, \quad A \in \mathcal{AH}_4. \tag{3.1}$$

In this vein, the norm should be written as

$$||\mathbf{A}|| = (\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A}), \tag{3.2}$$

the fourth degree radical

$$|\mathbf{A}| = \sqrt[4]{|(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{A})|} \tag{3.3}$$

will play the role of the length of the vector $\mathbf{A}$, and the corresponding symmetrical *quadralinear form* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ can easily be introduced such that, with respect to *the absolute basis*

$$\mathbf{S_1} = (1,0,0,0), \quad \mathbf{S_2} = (0,1,0,0), \quad \mathbf{S_3} = (0,0,1,0), \quad \mathbf{S_4} = (0,0,0,1) \tag{3.4}$$

(cf. (2.25)-(2.28)), one has

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = \frac{1}{4!}(a_1b_2c_3d_4 + a_1b_2c_4d_3 + \ldots + a_4b_3c_2d_1). \tag{3.5}$$

DEFINITION. The $\mathcal{AH}_4$-image (A, B, C, D) of the quadralinear form (**A,B,C,D**) is called *the scalar quadraproduct* in the $\mathcal{AH}_4$-algebra.

The form (3.5) shows the symmetry

$$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = (\mathbf{B}, \mathbf{A}, \mathbf{C}, \mathbf{D}) = (\mathbf{C}, \mathbf{B}, \mathbf{A}, \mathbf{D}) = \ldots (\mathbf{A}, \mathbf{B}, \mathbf{D}, \mathbf{C})$$

and the property of linearity:

$$(\mu\mathbf{A} + \nu\mathbf{E}, \mathbf{B}, \mathbf{C}, \mathbf{D}) = \mu(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}) + \nu(\mathbf{E}, \mathbf{B}, \mathbf{C}, \mathbf{D}), \ \ldots$$

(with respect to each argument), where $\mu, \nu \in \mathbb{R}$.

Using the initial quadralinear form (3.5), it is possible to construct in $\mathcal{VH}_4$ three mixed metric forms from two single vectors, proposing

$$(\mathbf{A}, \mathbf{A}, \mathbf{A}, \mathbf{B}) = \frac{1}{4}\left(a_1a_2a_3b_4 + a_1a_2a_4b_3 + a_1a_3a_4b_2 + a_2a_3a_4b_1\right), \tag{3.6}$$

$$(\mathbf{A}, \mathbf{A}, \mathbf{B}, \mathbf{B}) = \frac{1}{6}\left(a_1a_2b_3b_4 + a_1a_3b_2b_4 + a_1a_4b_2b_3 + a_2a_3b_1b_4 + a_2a_4b_1b_3 + a_3a_4b_1b_2\right), \tag{3.7}$$

and

$$(\mathbf{A}, \mathbf{B}, \mathbf{B}, \mathbf{B}) = \frac{1}{4}\left(a_1b_2b_3b_4 + a_2b_1b_3b_4 + a_3b_1b_2b_4 + a_4b_1b_2b_3\right). \tag{3.8}$$

In case of *unit vectors*,

$$\mathbf{a} = \frac{\mathbf{A}}{|\mathbf{A}|}, \quad \mathbf{b} = \frac{\mathbf{B}}{|\mathbf{B}|}, \quad \mathbf{c} = \frac{\mathbf{C}}{|\mathbf{C}|}, \quad \mathbf{d} = \frac{\mathbf{D}}{|\mathbf{D}|}, \tag{3.9}$$

so that

$$(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{a}) = (\mathbf{b},\mathbf{b},\mathbf{b},\mathbf{b}) = (\mathbf{c},\mathbf{c},\mathbf{c},\mathbf{c}) = (\mathbf{d},\mathbf{d},\mathbf{d},\mathbf{d}) = 1, \tag{3.10}$$

we get

$$(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) = \frac{(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B})}{|\mathbf{A}|^3|\mathbf{B}|}, \tag{3.11}$$

$$(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}) = \frac{(\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B})}{|\mathbf{A}|^2|\mathbf{B}|^2}, \tag{3.12}$$

$$(\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) = \frac{(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B})}{|\mathbf{A}||\mathbf{B}|^3}, \tag{3.13}$$

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \frac{(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})}{|\mathbf{A}||\mathbf{B}||\mathbf{C}||\mathbf{D}|}, \tag{3.14}$$

and

$$(\mathbf{a},\mathbf{a},\mathbf{a},\mathbf{b}) = \frac{1}{4}\left(\frac{a_1a_2a_3}{b_1b_2b_3} + \frac{a_1a_2a_4}{b_1b_2b_4} + \frac{a_1a_3a_4}{b_1b_3b_4} + \frac{a_2a_3a_4}{b_2b_3b_4}\right), \tag{3.15}$$

$$(\mathbf{a},\mathbf{a},\mathbf{b},\mathbf{b}) = \frac{1}{6}\left(\frac{a_1a_2}{b_1b_2} + \frac{a_1a_3}{b_1b_3} + \frac{a_1a_4}{b_1b_4} + \frac{a_2a_3}{b_2b_3} + \frac{a_2a_4}{b_2b_4} + \frac{a_3a_4}{b_3b_4}\right), \tag{3.16}$$

$$(\mathbf{a},\mathbf{b},\mathbf{b},\mathbf{b}) = \frac{1}{4}\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4}\right). \tag{3.17}$$

Given a vector $\mathbf{C} \in \mathcal{VH}_\mathbf{4}$, which is equal to geometric difference of two other vectors $\mathbf{A} \in \mathcal{VH}_\mathbf{4}$ and $\mathbf{B} \in \mathcal{VH}_\mathbf{4}$, we may use the numerical values of these metric forms to get for the difference measure

$$||\mathbf{C}|| = (\mathbf{C},\mathbf{C},\mathbf{C},\mathbf{C}) = (\mathbf{A}-\mathbf{B},\mathbf{A}-\mathbf{B},\mathbf{A}-\mathbf{B},\mathbf{A}-\mathbf{B}) \tag{3.18}$$

the generalized metric representation:

$$||\mathbf{C}|| = (\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) - 4(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B}) + 6(\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B}) - 4(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B}) + (\mathbf{B},\mathbf{B},\mathbf{B},\mathbf{B}) \tag{3.19}$$

Similarly as the cosine of an angle between two vectors in an Euclidean space allows to express the length of their geometric difference through the lengths of addends, we get

$$(\mathbf{C},\mathbf{C}) = (\mathbf{A}-\mathbf{B},\mathbf{A}-\mathbf{B}) = (\mathbf{A},\mathbf{A}) - 2(\mathbf{A},\mathbf{B}) + (\mathbf{B},\mathbf{B}) \equiv |\mathbf{A}|^2 - 2|\mathbf{A}||\mathbf{B}|(\mathbf{a},\mathbf{b}) + |\mathbf{B}|^2,$$

that is,

$$(\mathbf{C},\mathbf{C}) = |\mathbf{A}|^2 - 2|\mathbf{A}||\mathbf{B}|\cos(\mathbf{A},\mathbf{B}) + |\mathbf{B}|^2. \tag{3.20}$$

Therefore, the concept of hyperbolic functions can be extended to include the set of four *quadrahyperbolic cosines*:

$$\cosh(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B}), \quad \cosh(\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B}), \quad \cosh(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B}), \quad \cosh(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D}) \tag{3.21}$$

meaningful in the space $\mathcal{VH}_4$. These trigonometric functions are scalar functions of four unit vectors, or three real arguments, in contrast to the bilinear theories in which the trigonometric functions depend on but two vectors.

Finally, the pseudo-Euclidean concept of the pseudosphere (of the hyperboloid) can be extended according to

DEFINITION. The 3-dimensional hypersurface

$$\mathcal{I}_4 = \{\mathbf{A} \in \mathcal{VH}_4 : \, ||\mathbf{A}|| = 1, \, a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0\} \tag{3.22}$$

is called *the $\mathcal{H}_4$-hyperboloid*.

Generally, due analogs of the above definition should be applied to each of 16 sectors of the space $\mathcal{VH}_4$; we, however, restrict our treatment to the a_1'-oriented sector for definiteness, unless otherwise stated explicitly.

Extension of the pseudosphere of radius r should read

$$\mathcal{I}_4(r) = \{\mathbf{A} \in \mathcal{VH}_4 : \, ||\mathbf{A}|| = r > 0 \, , a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0\}. \tag{3.23}$$

This hypersurface may naturally be called *the $\mathcal{H}_4$-hyperboloid of radius r*. In accord with this terminology we may refer to (3.22) as to *the unit $\mathcal{H}_4$-hyperboloid*:

$$\mathcal{I}_4 = \mathcal{I}_4(1). \tag{3.24}$$

4. Transversality, Orthogonality, and Angle in $\mathcal{VH}_4$

Let us introduce

DEFINITION. A vector $\mathbf{B}$ is said *transversal* to a vector $\mathbf{A}$ if the metric form $(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B})$ is equal to zero. If additionally the alternative metric form $(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B})$ is equal to zero, then the vectors $\mathbf{A}$ and $\mathbf{B}$ are called *mutually transversal*.

The nature of a quadrahyperbalic space proves to be such that there are no mutually transversal unit vectors for which the metric form $(\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B})$ would also vanish. Instead they may equal its extreme values $+\frac{1}{3}$ or $-\frac{1}{3}$.

Let's adopt to call two nonisotropic vectors of a quadrahyperbolic space *orthogonal*, if the metric forms derivated therefrom obey the conditions

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B}) = 0, \quad (\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B}) = -\frac{1}{3}|\mathbf{A}|^2|\mathbf{B}|^2, \quad (\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B}) = 0. \tag{4.1}$$

In view of (3.18), this entails for two orthogonal vectors of quadrahyperbolic space the generalized representation

$$|\mathbf{C}|^4 = |\mathbf{A}|^4 - 2|\mathbf{A}|^2|\mathbf{B}|^2 + |\mathbf{B}|^4 = (|\mathbf{A}|^2 - |\mathbf{B}|^2)^2, \tag{4.2}$$

so that an analog of the Pythagorean theorem reads as

$$|\mathbf{C}|^2 = |\mathbf{A}|^2 - |\mathbf{B}|^2 \tag{4.3}$$

(here, the vectors $\mathbf{A}$ and $\mathbf{B}$ may belong to various sectors of the space $\mathcal{VH}_4$), which outwardly coincides with the expression for orthogonal vectors in a pseudo-Euclidean space.

If four noncoplanar vectors of a quadrahyperbolic space satisfy pairwisely the properties (4.1) and, additionally, have unit lengths, we say that the vectors comprise an orthonormal basis (which is a counterpart of an orthonormal basis in a bilinear space).

Whether the basic units 1, I, J, K, when changed for their bold counterparts $\mathbf{1}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, form an orthonormal basis of a quadrahyperbolic space under study? Since direct calculations yield the values

$$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) = 1, \quad (\mathbf{I},\mathbf{I},\mathbf{I},\mathbf{I}) = 1, \quad (\mathbf{J},\mathbf{J},\mathbf{J},\mathbf{J}) = 1, \quad (\mathbf{K},\mathbf{K},\mathbf{K},\mathbf{K}) = 1, \tag{4.4}$$

$$(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{I}) = (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{J}) = \ldots = (\mathbf{1},\mathbf{K},\mathbf{K},\mathbf{K}) = 0, \tag{4.5}$$

$$(\mathbf{1},\mathbf{1},\mathbf{I},\mathbf{I}) = (\mathbf{1},\mathbf{1},\mathbf{J},\mathbf{J}) = \ldots = (\mathbf{J},\mathbf{J},\mathbf{K},\mathbf{K}) = -\frac{1}{3}, \tag{4.6}$$

$$(\mathbf{1},\mathbf{I},\mathbf{J},\mathbf{J}) = (\mathbf{1},\mathbf{I},\mathbf{K},\mathbf{K}) = \ldots = (\mathbf{I},\mathbf{J},\mathbf{K},\mathbf{K}) = 0, \tag{4.7}$$

$$(\mathbf{1},\mathbf{I},\mathbf{J},\mathbf{K}) = \frac{1}{3}, \tag{4.8}$$

the answer to the above question should be said in positive.

Any other quadruples of vectors giving rise to the same values for metric forms may be summarizingly expressed by the following conditions:

$$(\mathbf{E_p},\mathbf{E_q},\mathbf{E_r},\mathbf{E_s}) = \begin{cases} 1 & if \quad p=q=r=s, \\ 0 & if \quad (p=q=r)\neq s, \\ -\frac{1}{3} & if \quad (p=q)\neq(r=s), \\ 0 & if \quad p=(q\neq r\neq s), \\ \frac{1}{3} & if \quad (p\neq q\neq r\neq s). \end{cases} \tag{4.9}$$

They will also make an orthonormal basis; the parentheses in the right-hand side mean all possible permutations.

With respect to the absolute basis $\mathbf{S_1},\mathbf{S_2},\mathbf{S_3},\mathbf{S_4}$, components of each vector $\mathbf{A}\in\mathcal{VH}_4$ are found to be $24(\mathbf{A},\mathbf{S_2},\mathbf{S_3},\mathbf{S_4})$, $24(\mathbf{S_1},\mathbf{A},\mathbf{S_3},\mathbf{S_4})$, $24(\mathbf{S_1},\mathbf{S_2},\mathbf{A},\mathbf{S_4})$, $24(\mathbf{S_1},\mathbf{S_2},\mathbf{S_3},\mathbf{A})$. Whence we arrive at the expansion

$$A = 24\Big((\mathbf{A},\mathbf{S_2},\mathbf{S_3},\mathbf{S_4})\mathbf{S_1} + (\mathbf{S_1},\mathbf{A},\mathbf{S_3},\mathbf{S_4})\mathbf{S_2} + (\mathbf{S_1},\mathbf{S_2},\mathbf{A},\mathbf{S_4})\mathbf{S_3} + (\mathbf{S_1},\mathbf{S_2},\mathbf{S_3},\mathbf{A})\mathbf{S_4}\Big). \tag{4.10}$$

If we go over to another basis $\mathbf{E_q}$, connected with an absolute one by means of the transformation $\mathbf{S_p} = M_{pq}\mathbf{E_q}$, the expansion of the vector $\mathbf{A}$ will modify to read

$$\mathbf{A} = 24((\mathbf{A}, M_{2q}\mathbf{E_q}, M_{3q}\mathbf{E_q}, M_{4q}\mathbf{E_q})M_{1q}\mathbf{E_q} + (M_{1q}\mathbf{E_q},\mathbf{A},M_{3q}\mathbf{E_q},M_{4q}\mathbf{E_q})M_{12q}\mathbf{E_q} \tag{4.11}$$

$$+ (M_{1q}\mathbf{E_q}, M_{2q}\mathbf{E_q},\mathbf{A},M_{4q}\mathbf{E_q})M_{3q}\mathbf{E_q} + (M_{1q}\mathbf{E_q},M_{2q}\mathbf{E_q},M_{3q}\mathbf{E_q},\mathbf{A})M_{4q}\mathbf{E_q}). \tag{4.12}$$

In particular, for the orthonormalized basis $\mathbf{1}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ the remarkable representation

$$\mathbf{A} = (\mathbf{A},\mathbf{1},\mathbf{1},\mathbf{1})\mathbf{1} + (\mathbf{A},\mathbf{I},\mathbf{I},\mathbf{I})\mathbf{I} + (\mathbf{A},\mathbf{J},\mathbf{J},\mathbf{J})\mathbf{J} + (\mathbf{A},\mathbf{K},\mathbf{K},\mathbf{K})\mathbf{K} \tag{4.13}$$

can be obtained after simple direct calculations. Thus, the forms $(\mathbf{A},\mathbf{1},\mathbf{1},\mathbf{1})$, $(\mathbf{A},\mathbf{I},\mathbf{I},\mathbf{I})$, $(\mathbf{A},\mathbf{J},\mathbf{J},\mathbf{J})$, and $(\mathbf{A},\mathbf{K},\mathbf{K},\mathbf{K})$ assign the connection of vector $\mathbf{A}$ components in an orthonormalized basis with the product $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$, and therefore these forms can be interpreted as *transversal projections* of the given vector on the axes $\mathbf{1}$, $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$.

Four respective transversal projections of a unit vector $\mathbf{a} \in \mathcal{VH}_4$ on an orthonormal basis's axes prove to be connected by the following relation

$$\cosh^4(\mathbf{a},\mathbf{1},\mathbf{1},\mathbf{1}) + \cosh^4(\mathbf{a},\mathbf{I},\mathbf{I},\mathbf{I}) + \cosh^4(\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{J}) + \cosh^4(\mathbf{a},\mathbf{K},\mathbf{K},\mathbf{K})$$

$$-2\cosh^2(\mathbf{a},\mathbf{1},\mathbf{1},\mathbf{1})\cosh^2(\mathbf{a},\mathbf{I},\mathbf{I},\mathbf{I}) - 2\cosh^2(\mathbf{a},\mathbf{1},\mathbf{1},\mathbf{1})\cosh^2(\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{J})$$

$$-2\cosh^2(\mathbf{a},\mathbf{1},\mathbf{1},\mathbf{1})\cosh^2(\mathbf{a},\mathbf{K},\mathbf{K},\mathbf{K}) - 2\cosh^2(\mathbf{a},\mathbf{I},\mathbf{I},\mathbf{I})\cosh^2(\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{J})$$

$$-2\cosh^2(\mathbf{a},\mathbf{I},\mathbf{I},\mathbf{I})\cosh^2(\mathbf{a},\mathbf{K},\mathbf{K},\mathbf{K}) - 2\cosh^2(\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{J})\cosh^2(\mathbf{a},\mathbf{K},\mathbf{K},\mathbf{K})$$

$$+\,8\cosh(\mathbf{a},\mathbf{1},\mathbf{1},\mathbf{1})\cosh(\mathbf{a},\mathbf{I},\mathbf{I},\mathbf{I})\cosh(\mathbf{a},\mathbf{J},\mathbf{J},\mathbf{J})\cosh(\mathbf{a},\mathbf{K},\mathbf{K},\mathbf{K}) = 1, \tag{4.14}$$

which, as a matter of fact, is an analog of Pythagorean theorem and extend the latter for a unit diagonal of a quadrahyperbolic parallelotope which edges are pairwisely orthogonal.

By an $\mathcal{H}_4$-*angle* ϕ between two vectors $\mathbf{A}$ and $\mathbf{B}$ we shall naturally mean the length of a geodesic arc which joins over the unit $\mathcal{H}_4$-hyperboloid $\mathcal{I}_4$ the points of intersection of the vectors (or their straight continuations) with $\mathcal{I}_4$. It is possible to show that its numerical value is determined by

$$\phi \stackrel{\text{def}}{=} \Big|\ln(\mathbf{A}) - \ln(\mathbf{B})\Big| = \Big|(\alpha_A - \alpha_B)\mathbf{I} + (\beta_A - \beta_B)\mathbf{J} + (\gamma_A - \gamma_B)\mathbf{K}\Big|, \tag{4.15}$$

where $\ln(\mathbf{A})$ means the logarithmic function (2.42) of the quadrahyperbolic variable A that relates to the vector $\mathbf{A}$. We remain it to the reader to verify the validity of the component representation

$$\phi = \sqrt[4]{\left|\ln\frac{a_1}{b_1}\ln\frac{a_2}{b_2}\ln\frac{a_3}{b_3}\ln\frac{a_4}{b_4}\right|}, \tag{4.16}$$

which implies the relations $a_1a_2a_3a_4 = 1$ and $b_1b_2b_3b_4 = 1$.

5. Arcs, $\mathcal{VH}_4$-Sectors, $\mathcal{VH}_4$-Cones, and Isometries

We introduce

DEFINITION. Given two points on the $\mathcal{H}_4$-hyperboloid (3.22), the geodesic piece that joins the points is called an *arc*.

DEFINITION. A two-dimensional surface formed by two vectors $\mathbf{A}$ and $\mathbf{B}$ subject to the condition that the ends of the vectors are connected by an arc will be called *the*

$\mathcal{VH}_4$-*sector*, to be denoted as $\mathcal{T}_4(\mathbf{A},\mathbf{B})$. If the vectors are unit, the adjective *unit* will be added to the names of these sectors.

This expands the corresponding concept used in bilinear spaces, where triangles and sectors are always plane figures; a $\mathcal{VH}_4$-sector is generally not a plane figure, - being rather a "cone-type surface".

DEFINITION. The $\mathcal{VH}_4$-*cone* $\mathcal{C}_4(r)$ is a two-dimensional surface which generatrix is a semiline issued from the origin of the space $\mathcal{VH}_4$ and which intersection with the unit $\mathcal{H}_4$-hyperboloid $\mathcal{I}_4$ is an r-radius circle drawn on the $\mathcal{I}_4$.

The latter definition provides an extension of the ordinary Euclidean cone of rotation.

DEFINITION. Two figures formed by a set of vectors of equal number are called *isometric* if all respective scalar products, in the sense of the definitions introduced by the list (3.6)-(3.8), have equal values.

In accordance with the latter definition, given two pairs of vectors of a quadralinear space, $\mathbf{A}$ and $\mathbf{B}$, *resp.* $\mathbf{A}'$ and $\mathbf{B}'$, the figure made up by the first two vectors will be isometric to the figure made up by the last two figures, when all the equalities

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = (\mathbf{A}',\mathbf{A}',\mathbf{A}',\mathbf{A}'), \tag{5.1}$$

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{B}) = (\mathbf{A}',\mathbf{A}',\mathbf{A}',\mathbf{B}'), \tag{5.2}$$

$$(\mathbf{A},\mathbf{A},\mathbf{B},\mathbf{B}) = (\mathbf{A}',\mathbf{A}',\mathbf{B}',\mathbf{B}'), \tag{5.3}$$

$$(\mathbf{A},\mathbf{B},\mathbf{B},\mathbf{B}) = (\mathbf{A}',\mathbf{B}',\mathbf{B}',\mathbf{B}'), \tag{5.4}$$

and

$$(\mathbf{B},\mathbf{B},\mathbf{B},\mathbf{B}) = (\mathbf{B}',\mathbf{B}',\mathbf{B}',\mathbf{B}') \tag{5.5}$$

hold simultaneously. In particular, two vectors $\mathbf{A}$ and $\mathbf{A}'$ of a quadralinear space are isometric if

$$(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A}) = (\mathbf{A}',\mathbf{A}',\mathbf{A}',\mathbf{A}') \neq 0. \tag{5.6}$$

6. $\mathcal{H}_4$-Holomorphic Functions

Given *a function of quadrahyperbolic variable*:

$$f(A) = U(a'_1,a'_2,a'_3,a'_4) + V(a'_1,a'_2,a'_3,a'_4)I + W(a'_1,a'_2,a'_3,a'_4)J + Q(a'_1,a'_2,a'_3,a'_4)K, \tag{6.1}$$

where U,V,W,Q are smooth functions of four real arguments, we treat the set U,V,W,Q naturally as *the hypercomplex components* of the function f.

DEFINITION. The function (6.1) is called $\mathcal{H}_4$-*holomorphic*, if the following $\mathcal{H}_4$-*holomorphic conditions*

$$\frac{\partial U}{\partial a'_1} = \frac{\partial V}{\partial a'_2} = \frac{\partial W}{\partial a'_3} = \frac{\partial Q}{\partial a'_4}, \qquad \frac{\partial U}{\partial a'_2} = \frac{\partial V}{\partial a'_1} = \frac{\partial W}{\partial a'_4} = \frac{\partial Q}{\partial a'_3}, \tag{6.2}$$

$$\frac{\partial U}{\partial a'_3}=\frac{\partial V}{\partial a'_4}=\frac{\partial W}{\partial a'_1}=\frac{\partial Q}{\partial a'_2}, \qquad \frac{\partial U}{\partial a'_4}=\frac{\partial V}{\partial a'_3}=\frac{\partial W}{\partial a'_2}=\frac{\partial Q}{\partial a'_1} \tag{6.3}$$

hold fine.

To elucidate the meaning of the latter conditions, let us consider the direct differentials

$$\Delta A=\Delta a'_1 1+\Delta a'_2 I+\Delta a'_3 J+\Delta a'_4 K \tag{6.4}$$

and

$$\Delta f=\Delta U+\Delta V I+\Delta W J+\Delta Q K, \tag{6.5}$$

and introduce the partial derivative

$$\partial f=\{\partial_a f\}: \tag{6.6}$$

$$\partial_1 f=\frac{\partial U}{\partial a'_1}+\frac{\partial V}{\partial a'_1}I+\frac{\partial W}{\partial a'_1}J+\frac{\partial Q}{\partial a'_1}K, \tag{6.7}$$

$$\partial_2 f=\frac{\partial U}{\partial a'_2}+\frac{\partial V}{\partial a'_2}I+\frac{\partial W}{\partial a'_2}J+\frac{\partial Q}{\partial a'_2}K, \tag{6.8}$$

$$\partial_3 f=\frac{\partial U}{\partial a'_3}+\frac{\partial V}{\partial a'_3}I+\frac{\partial W}{\partial a'_3}J+\frac{\partial Q}{\partial a'_3}K, \tag{6.9}$$

$$\partial_4 f=\frac{\partial U}{\partial a'_4}+\frac{\partial V}{\partial a'_4}I+\frac{\partial W}{\partial a'_4}J+\frac{\partial Q}{\partial a'_4}K. \tag{6.10}$$

Now we want to retain the ordinary multiplication rule

$$\Delta f=\Delta A\cdot\partial f. \tag{6.11}$$

It can readily be verified that *Eqs.* (6.4)-(6.11) *entail the $\mathcal{H}_4$-conditions* (6.2)-(6.3).

7. Relationship with Finsler Geometry

We may identify the Finslerian metric function

$$F(\mathbf{A})=\sqrt[4]{|(\mathbf{A},\mathbf{A},\mathbf{A},\mathbf{A})|} \tag{7.1}$$

with the length (3.3). Owing to (2.40), we can use the component representation

$$F(\mathbf{A})=\sqrt[4]{|a_1a_2a_3a_4|} \tag{7.2}$$

which is identical with the known representation of the *Berwald-Moor* metric function (see [25]). If we construct the associated covariant components

$$y_p \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial F^2}{\partial a^p}, \tag{7.3}$$

where $a^p = \{a_1, a_2, a_3, a_4\}$, we get

$$y_p = \frac{F^2}{4a^p}. \tag{7.4}$$

Farther calculation of the Finslerian metric tensor

$$g_{pq} \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 F^2}{\partial a^p \partial a^q} \tag{7.5}$$

yields

$$g_{pq} = \frac{2y_p y_q}{F^2} - \frac{F^2}{4a^p a^q}\delta_{pq}, \tag{7.6}$$

where δ stands for the Kronecker symbol.

By comparing (2.24) with (7.4) we may conclude that the Finslerian concept of covariant components is tantamount to the concept of the inverse number in the quadralinear space.

Also, *the Finslerian indicatrix* given by

$$F(\mathbf{A}) = 1 \tag{7.7}$$

is equivalent to the $\mathcal{H}_4$-hyperboloid (3.22). Many other analogies can be traced farther.

8. Conclusions and Prospects

Various significant cases of polyspaces can be specified to make interesting applications. Under quadralinear symmetric-type treatment, the fundamental metric forms

$$||A|| = {a_1}^4 + {a_2}^4 + {a_3}^4 + {a_4}^4, \tag{8.1}$$

$$||A|| = (a_2 + a_3 + a_4){a_1}^3 + (a_1 + a_3 + a_4){a_2}^3 + (a_1 + a_2 + a_4){a_3}^3 + (a_1 + a_2 + a_3){a_4}^3, \tag{8.2}$$

$$||A|| = {a_1}^2{a_2}^2 + {a_1}^2{a_3}^2 + {a_1}^2{a_4}^2 + {a_2}^2{a_3}^2 + {a_2}^2{a_4}^2 + {a_3}^2{a_4}^2, \tag{8.3}$$

$$||A|| = (a_2a_3 + a_2a_4 + a_3a_4){a_1}^2 + (a_1a_3 + a_1a_4 + a_3a_4){a_2}^2$$

$$+ (a_1a_2 + a_1a_4 + a_2a_4){a_3}^2 + (a_1a_2 + a_1a_3 + a_2a_3){a_4}^2, \tag{8.4}$$

$$||A|| = a_1a_2a_3a_4 \tag{8.5}$$

seem to be most attractive because of their simple algebraic structures. In the present work, we have opened up due possibilities inhereted in the last choice (8.5). Obviously, the above forms can be considered as "canonical ingredients" for farther classification of quadralinear spaces. The forms (8.1)-(8.5) are algebraically independent, - in the sense that no member of the list (8.1)-(8.5) can be mapped under a linear transformation in another member of this list. Therefore, the forms may serve as canonical elements for classification of quadralinear spaces; extension to any two-, three-, or ($n > 4$)-dimensional theories is straightforward in many aspects.

Because of the apparent mathematical simplicity and beauty, and also rather non-trivial geometric structure of the polyspaces considered above, it can be hoped that the content and subject of the present paper may be of interest also to those who develop and study new physical generalized-relativisitic approaches rather than proper algebra of polynumbers.

Research Problems

There is much interesting work still to be done on the ideas and methods proposed above. The following list of nearest pending problems may be set forth to resolve, - which way, whenever being successful, would favour various practical as well as relativistic scopes of applications of commutative and associative hypercomplex numbers and multilinear spaces treated above.

1. Develop an appropriate systematization for polynumbers of square-type nature. Classify polynumbers which fail to be of square-type nature.

2. Offer a simple algorithm of constructing mutual-conjugates to match a maximally broad set of polynumbers.

3. Propose self-consistent geometric treatment of arguments α, β, γ (see Eqs. (2.44)-(2.49)) of exponential expression of quadrahyperbolic (and other) polynumbers. Do the arguments admit a meaning of geometrical angles proper?

4. Find approporiate extended $\mathcal{H}_4$-rotation transformations. Can the sectors $\mathcal{T}_4$ or the cones $\mathcal{C}_4$ in the $\mathcal{VH}_4$-space be moved subject to the condition that the lengths of generating vectors and arcs remain unchanged?

5. Trace the possibilities to have congruent transformations in the spaces under study. Is it possible to find the congruent transformations such that isometric figures (see the conditions (5.1)-(5.5)) can be moved to identify?

6. In multilinear spaces, define additive metric parameters for figures made up of three and more vectors.

7. Propose and develop, up to isomorphism, classification of four-dimensional spaces with quadralinear symmetrical forms.

8. For the spaces under study, investigate existence of two-dimensional hyperplanes, as well as three-dimensional hypersurfaces, such that they are endowed with a Riemannian-type metric.

9. Ellucidate the structure of holomorphic functions of polynumbers and develop respective geometric treatments. Study singular points, lines, and regions appeared under analitic mappings of associated hypercomplex manifolds.

10. Describe due extensions for arbitrary dimensions.

11. Seek to develop novel generalized physical aspects, in particular those referred to light behaviour, basig on the $\mathcal{AH}_4$-algebra or on the $\mathcal{VH}_4$-space.

12. Understand and investigate connection of Pythagorean theorem analogs for length of diagonals of parallelotopes in multilinear spaces with Diophantine equations.

References

[1] H. Helmholtz: *Ueber die Thatsachen die der Geometrie zum Grunde liegen,* Göttingen, Nachr. 1868.

[2] I.L. Kantor, A.S. Solodovnikov: *Hypercomplex numbers: An elementary introduction to algebras*, Springer, Berlin 1989.

[3] G. Birkhoff and S. MacLane: *Modern Algebra,* Macmillan, New York, Third Edition 1965.

[4] B.L. van der Waerden: *Modern Algebra,* F.Ungar, New York, Third Edition 1950.

[5] O. Taussky: *Algebra,* in Handbook of Physics, ed. by E. U.Condon and H. Odishaw, McGraw-Hill, New York, Second Edition 1958.

[6] J.W. Brown and R.V.Churchill: *Complex variables and applications,* McGraw-Hill, New York, 1996.

[7] R.L. Goodstein: *Complex functions,* McGraw-Hill, New York, 1965.

[8] K. Gürlebeck and W.Sprössig: *Quaternionic and Clifford calculus for physicists and engineers,* John Wiley&Sons, 1997.

[9] G.B. Price: *An introduction to multicomplex spaces and functons,* Marcel Dekker, 1991.

[10] J.B. Seaborn: *Hypergeometric functions and their applications,* Springer, Berlin 1991.

[11] A. van Proeyen: Special geometries, from real to quaternionic, hep-th/0110263.

[12] Y. Nutku, M.B. Sheftel: A family of heavenly metrics, gr-qc/0105088.

[13] G. Papadopoulos: Brane Solitons and Hypercomplex Structures, math.DG/0003024.

[14] D.M.J. Calderbank and P. Tod: Einstein metrics, hypercomplex structures and the Toda field equation, math.DG/9911121.

[15] M. Ansorg: Differentially rotating disks of dust: Arbitrary rotation law, gr-qc/0006045.

[16] S. Ollariu: Complex numbers in n dimension, math.CV/9011077.

[17] D.M. J. Calderbank, Einstein metrics, hypercomplex structures and the Toda field equation, P. Tod, math.DG/9911121.

[18] D.L. Stefano: Hypercomplex group theory, physics/9703033.

[19] A. Dimakis and F. Müller-Hoissen: Bicomplex and Bäcklund transformations, nlin.SI/0104071.

[20] D. Joyce: A theory of quaternionic algebra, with applications to hypercomplex geometry, math.DG/0010079.

[21] S. Rönn: Bicomplex algebra and function theory, math.CV/0101200.

[22] D.L. Stefano and R. Pietro: Local Hypercomplex Analyticity, funct-an/9703002.

[23] A. Sudbery: Quaternionic analysis, *Math. Proc. Camb. Phil. Soc.* **85** (1979), 199.

[24] H. Rund: *The Differential Geometry of Finsler Spaces,* Springer, Berlin 1959.

[25] G.S. Asanov: *Finsler Geometry, Relativity and Gauge Theories,* D. Reidel Publ. Comp., Dordrecht 1985.

[26] G.S. Asanov: Finsleroids reflect future-past asymmetry. *Rep. Math. Phys.* **47** (2001), 323.

[27] G.S. Asanov: Can neutrinos and high-energy particles test Finsler metric of space-time? hep-ph/0009305.

[28] G.S. Asanov: Finslerian extension of Lorentz transformations and first-order censorship theorem. *Found. Phys. Lett.* **15** (2002), 199.

[29] G.S. Asanov: Finslerian anisotropic relativistic metric function obtainable under breakdown of rotational symmetry. gr-qc/0204070.